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ANALYSIS OF LEAD TIME ON PERMISSIBLE DELAY IN PAYMENTS IN AN INVENTORY MODEL INCLUDING THE LEAD TIME CRASHING COST

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Abstract. In this article, the lead time on permissible delay in payments in an inventory model including lead time crashing cost is discussed where lead time and business period are the decision variables. Also, the lead time dependent credit period has been considered which has two parts one being fixed and other being dependent upon lead time. Here supplier offers the credit period to the retailer only when supplier supplies the order before the end of the business period. Here model has been considered in the parlance of infinite time horizon in such a way that the system gets the maximum profit. There are two main cases of inventory models to be studied here. Finally, three different illustrative examples have been added to determine the optimal policy of the model and the sensitivity analysis of some parameters has been added in this model.

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1 Introduction

Lead time is the time that elapses between the placement of an order and the receipt of the order into inventory. Lead time may influence customer service and impact inventory costs. From the literature, it is known that productivity of the company and its competitive position in the market depends on lead time. Traditionally, in an inventory models, minimization of total cost or, maximization of total profit has been considered as an objective function from either the supplier's or manufacturer's/retailer's side. In 1975, Das (1975) stated the effect of lead time on inventory and give a static analysis about it. Foote et al. (1988) presented a heuristic policies for inventory ordering problems with long and randomly varying lead times. Ouyang and Wu (1998) established a minimax distribution free procedure for mixed inventory model with variable lead time. Ben-Daya and Raouf (1994) presented an inventory models involving lead time as a decision variable. Glock (2012) discussed the inventory model in which customer service and responsiveness to production schedule changes can be improved by reduced lead time and reduction in safety stocks can be achieved. Hsiao (2008), He et al. (2005), Lan et al. (1999), Yang et al. (2005), Pan et al. (2004) stated that fixed lead time is not always appropriate for all inventory model in business, so they considered lead time as a decision variable. These authors have presented models which can be used to determine the length of lead time that minimizes the expected total relevant cost. Chopra et al. (2004) observed the effects of lead time uncertainty on safety stocks. Ouyang et al. (2004), Chang et al. (2006), and Wu (2004) presented the models with controllable lead time, in which they minimized the expected total relevant cost. Ray and Jewkes (2004) discussed a model in which demand and price both are lead time sensitive. We have provided a review table, which contains the summary of main literature (see Table 1).

Table 1: Summary of related literature for EOQ models with lead time

Author(s)	Constant demand	Lead time	Credit period	Lead time crashing cost as an exponential function	Lead time linked credit period
Yang et al. (2005)		√			
Chang et al. (2006)		√			
Liao (2007a)			√		
Hsiao (2008)		√			
Glock (2012)		√			
Guria et al. (2012)			√		
Musa and Sani (2012)			√		
Das et al. (2013)			√		
Uthayakumar and Priyan (2013)		√	√		
Das et al. (2014)					
Yang and Tseng (2014)		√	√		
Das et al. (2015)					
Banu and Mondal (2016)			√		
Present paper	√	√	√	√	√

Credit period is the time to delay the payment. When the account has settled, the customer can sell the goods and continues to accumulate revenue and earn interest if the supplier requires settlement of the account after replenishment. Interest earned can be thought of as a return on investment since the money generated through revenue can be ploughed back into the business. There fore, it makes economic sense for the customer to delay the settlement of the replenishment account up to the last day of the credit period allowed by supplier. If the credit period is less than the cycle length, the customer continues to accumulate revenue and earn interest on it for the rest of the period in the cycle, from the stock remaining beyond the credit period. This point was not considered by Goyal (1985). Purchasing cost and opportunity cost has been saved by taking the benefit of trade credit period, which is very important for deteriorating items. Liao (2008) and Guria et al. (2012) presented an *EOQ* model for deteriorating items under two-level trade credit and Das et al. (2013) have shown the impacts of credit periods on integrated inventory control systems. Aggarwal and Jaggi (1995), Liao (2007a), Musa and Sani (2012) and Huang (2007) worked on permissible delay for deteriorating items. Liao (2007b) presented an EOQ model for deteriorating items under supplier credit linked to ordering quantity. Das et al. (2014a) discussed about an integrated inventory model with delay in payment for deteriorating item under Weibull distribution and advertisement cum price-dependent demand. Guria et al. (2013) presented an inventory model with part payment, Das et al. (2014b) worked on an inventory model with a discrete credit period. Modal and Maiti (2003) have been worked on fuzzy EOQ models using genetic algorithm, Das et al. (2015) and Banu and Mondal (2016) presented an inventory model under interactive fuzzy credit period for deteriorating item. Considering lead time and back-order Yang and Tseng (2014) introduced three-echelon inventory model with permissible delay in payments. Uthayakumar and Priyan (2013) presented the two-echelon inventory system with controllable setup cost with permissible delay in payments and lead time under service level constraint. Vijayashree and Uthayakumar (2016) discussed an inventory models involving lead time crashing cost as an exponential function.

In this paper, first time, an attempt has been made to develop an inventory model in which a relationship between lead time and credit period has been considered in infinite time horizon. A crashing cost has been taken relating with lead time. Finally, the model has been optimized analytically to get the optimum expected profit by variable lead time under a permissible delay in payments. At last, to get the feasibility of the proposed model some numerical examples have been considered.

The content of the section is organized as follows: Section 2 presents 'Notations and Assumptions' of this model. In Section 3, the 'Mathematical Formation' of the model and in Section 4, 'Solution Procedure' of the model have been discussed. Some numerical examples and managerial implications have been viewed in Section 5. Finally a concluding remarks and acknowledgement are presented in Section 6 and Section 7 respectively.

2 Notations and Assumptions

To develop the proposed model, the following notations and assumptions have been used.

2.1 Notations

For convenience, the following notations are used throughout the entire paper.

L	:	The lead time in the business.
R	:	The re-order point in the inventory.
A_r	:	The ordering cost of the retailer per order.
h_r	:	The holding cost of retailer per unit quantity/ unit time.
Q	:	The order quantity.
S_r	:	The selling price of the retailer per unit item.
C_{pr}	:	The purchasing cost of the retailer per unit item.
T	:	The business period.
D	:	Demand per unit time.
M	:	Offered credit period by the supplier to the retailer.
M_0	:	Credit period when the retailer receives the order at time T from supplier.
$\omega(L)$	:	The lead time crashing cost.
α, β, γ	:	Parameters involved in credit period and crashing cost.
$EAC(T, L)$	:	The expected average total cost.
$\Pi(T, L)$	:	The expected average total profit.
I'_p	:	Rate interest paid per rupee investment in stocks per unit time.
I'_e	:	Rate interest earned per rupee per unit time.
$IPC^{(\gamma\neq 0)}$	:	Interest payable cost when supplier offers extra credit period to retailer.
$IPC^{(\gamma=0)}$	:	Interest payable cost when retailer does not accept the extra credit period.
$IE^{(\gamma\neq 0)}$	:	Interest earned when supplier offers extra credit period.
$IE^{(\gamma=0)}$	:	Interest earned when retailer does not accept the extra credit period.
EHC	:	The extra holding cost.

2.2 Assumptions

To develop the proposed model the following assumptions have been used.

(i) Inventory is continuously reviewed. Replenishment are made whenever the inventory level falls to the re-order point.

(ii) Shortages are not allowed.

(iii) The demand (D) is constant.

(iv) Lead time (L) is considered as a variable.

(v) Presently, in the competitive commercial market many suppliers and retailers would like to make a long term co-operative relationship. In this regard, the suppliers offer a credit period facility to his/her retailers to give some financial support to continue the business. Again, every retailer is keen to get delivery of goods from the supplier as soon as his/her stock is end in the business concern, due to reduce the holding cost. But, practically it is not possible often to deliver in proper time. So in most of times the supplier supplies the order before the end of business period. In this case, the retailer must carry an extra holding cost. For this situation, to give the advantage to the retailer in the credit period facility, the supplier varies the credit period on the basis of the delivery i.e., the supplier offer lead time dependent credit period facility. In this paper it has been considered in the following way

$$M = M_0 + \gamma\Big(\frac{R - DL}{D}\Big), \tag{1}$$

where $R > DL$, M_0 is the credit period when supplier supply the order at business period (T) and γ is a positive constant.

(vi) It is assumed that the offered credit period (M) by the supplier to the retailer be less or greater than equal to business period (T).

(vii) It is assumed that the supplier supply the order before the end of the business period (T).

($viii$) To develop the model, the crashing cost $\omega(L)$ has been considered which is related to the lead time by the following functional form

$$\omega(L) = \alpha e^{-\beta L},$$

where α and β are known as the effectiveness parameters of crashing cost. Here crashing is a method for shortening the duration of lead time by reducing the normal length of delivery time of the business. In most of the literature review in inventory problems, lead time is viewed as a prescribed constant or a stochastic variable. But, in numerous sensible circumstances, lead time can be reduced by an additional crashing cost. So, crashing cost has been assumed to reduce the lead time. This function shows that, lead time and crashing cost are inversely proportional. i.e., if L be very large, then there is less crashing cost.

3 Mathematical Formation of the model

In this model, a retailer starts his/her business at $t = 0$ and receives Q amount of items from the supplier. The retailer orders same amount of items again, when inventory level drops to the re-order point R. The re-order point R has been considered as
R=expected demand during lead time + safety stock(SS) , i.e., $R = DL + SS$.

In this proposed model, it has been considered that the supplier offers a credit period to the retailer accordingly to his/her policy. Since here, lead time has been considered, obviously the credit period should have dependency on lead time, which has been discussed in assumption (v). Generally, in many practical situation it is seen that the lead time can be reduced by adding a crashing cost. Hence forth, in this paper it has been considered according to assumption ($viii$). Now, the length of each cycle is T. Let $I(t)$ be the inventory level at any time t, $(0 \le t \le T)$. The differential equation that describes the instantaneous state of $I(t)$ over $(0, T)$ is given by

$$\frac{dI(t)}{dt} = -D,\ 0 \le t \le T\ .$$

The solution of the above differential equation with boundary conditions $I(0) = Q$ and $I(T) = 0$ is given by

$$I(t) = Q - Dt,\ 0 \le t \le T \ \text{ and } Q = DT\ .$$

Sales revenue (SR) for each cycle $= S_r \int_0^T D dt = S_r DT$

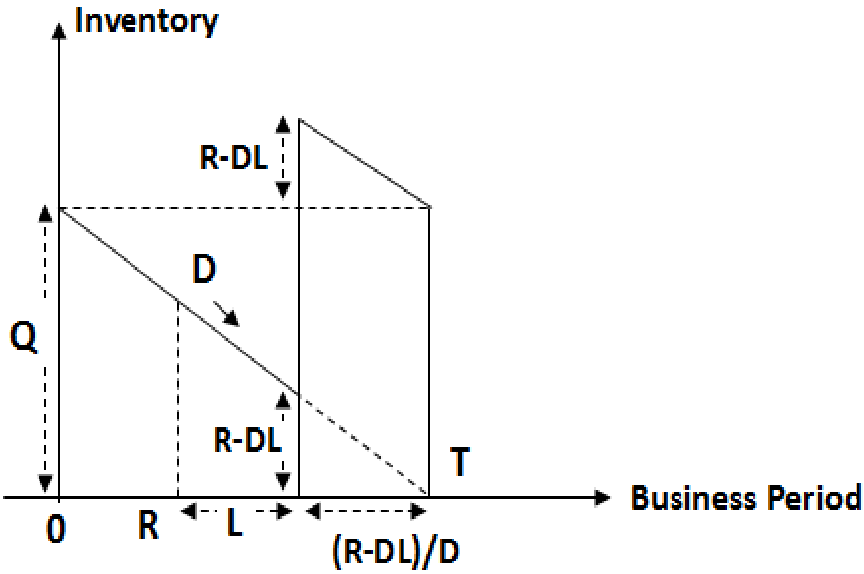

Figure 1. The inventory model

Including lead time crashing cost according to assumption $(viii)$, the expected average total cost is given by

$$\begin{aligned} EAC(T,L) &= \text{Average ordering cost + average purchasing cost + average holding cost} \\ &+ \text{average lead time crashing cost} \\ &= \frac{A_r}{T} + C_{pr}D + h_r\Big(\frac{1}{2}DT + R - DL\Big) + \frac{\alpha e^{-\beta L}}{T}. \end{aligned} \quad (2)$$

Now, according to the assumption (v), the offered credit period (M) by the supplier to the retailer, depends on the lead time. So depending on the values of L, the values of M may be greater than or less than the business period T. Therefore there may occur two cases, namely (i) $M \leq T$ and (ii) $M > T$.

3.1 When the credit period to be less than or equal to the business period $(M \leq T)$

In this case, retailer carry an interest payable cost per each cycle, because he/she pays the payment before the end of the business period. Also, the retailer earned interest per each cycle after settlement of his/her replenishment.
Now, interest payable per cycle is given by

$$C_{pr}I'_p\int_M^T I(t)dt = C_{pr}I'_p\Big[Q(T-M) - \frac{D}{2}(T^2 - M^2)\Big]$$

and interest earned per cycle is given by

$$S_rI'_e\int_0^T D(T-t)dt = S_rI'_e\Big(\frac{DT^2}{2}\Big).$$

Therefore, the expected average total profit $\Pi(T,L)$ is given by

$$\begin{aligned} \Pi(T,L) &= \Big(S_rD + \frac{S_rI'_eDT}{2}\Big) - \Big[\frac{A_r}{T} + C_{pr}D + h_r(\frac{1}{2}DT + R - DL) + \frac{\alpha e^{-\beta L}}{T} \\ &+ CI'_p\Big\{D(T-M) - \frac{D}{2}\Big(T - \frac{M^2}{T}\Big)\Big\}\Big]. \end{aligned} \quad (3)$$

Theorem 1. *If $M \leq T$, the profit function $\Pi(T, L)$ will be optimal for optimum value of $T = T^*$ and $L = L^*$ satisfying the relations*

$$\beta T^2(C_{pr}I'_pD + h_rD - S_rI'_eD) + 2(h_rD - C_{pr}I'_pD\gamma)T - \beta C_{pr}I'_pD.$$
$$\Big\{(M_0 + \frac{\gamma R}{D}) - \gamma L\Big\}^2 - 2A_r\beta + 2C_{pr}I'_pD\gamma(M_0 + \frac{\gamma R}{D} - \gamma L) = 0, \quad \textit{and}$$
$$(2A_r\alpha\beta^2e^{-\beta L} + 2A_rC_{pr}I'_pD\gamma^2 + \alpha^2\beta^2e^{-2\beta L} + 2\alpha C_{pr}I'_pD\gamma^2e^{-\beta L} + C_{pr}I'_pDM^2\alpha\beta^2e^{-\beta L}$$
$$- \; 2C_{pr}I'_pDM\alpha\beta\gamma e^{-\beta L}) > 0, \; \textit{then } \Pi(T^*, L^*) \textit{ attains global optimum value.}$$

Proof. Here the objective function of the model is given by

$$\begin{aligned}\Pi(T, L) \;\; &= \;\; \Big(S_rD + \frac{S_rI'_eDT}{2}\Big) - \Big[\frac{A_r}{T} + C_{pr}D + h_r(\frac{1}{2}DT + R - DL) + \frac{\alpha e^{-\beta L}}{T} \\ &+ \;\; C_{pr}I'_p\Big\{D(T - M) - \frac{D}{2}\Big(T - \frac{M^2}{T}\Big)\Big\}\Big].\end{aligned}$$

For optimal value of T and L, $\frac{\partial \Pi}{\partial T} = 0$ and $\frac{\partial \Pi}{\partial L} = 0$ give the following relations

$$\frac{\alpha e^{-\beta L}}{T} = T\Big(\frac{C_{pr}I'_pD}{2} + \frac{h_rD}{2}\Big) - \frac{C_{pr}I'_pDM^2}{2T} - \frac{S_rI'_eDT}{2} - \frac{A_r}{T} \tag{4}$$

$$\text{and } h_rD + \frac{\alpha\beta e^{-\beta L}}{T} - C_{pr}I'_pD\gamma + \frac{C_{pr}I'_pDM\gamma}{T} = 0. \tag{5}$$

Eliminating $\frac{\alpha e^{-\beta L}}{T}$ from equation (4) and (5), the following is obtained

$$\begin{aligned}&\beta T^2(C_{pr}I'_pD + h_rD - S_rI'_eD) + 2(h_rD - C_{pr}I'_pD\gamma)T - \beta C_{pr}I'_pD \\ &\Big\{(M_0 + \frac{\gamma R}{D}) - \gamma L\Big\}^2 - 2A_r\beta + 2C_{pr}I'_pD\gamma\Big(M_0 + \frac{\gamma R}{D} - \gamma L\Big) = 0\end{aligned} \tag{6}$$

Also, $(\frac{\partial^2 \Pi}{\partial T^2}).(\frac{\partial^2 \Pi}{\partial L^2}) - (\frac{\partial^2 \Pi}{\partial T \partial L})^2 > 0$ gives

$$\begin{aligned}&(2A_r\alpha\beta^2e^{-\beta L} + 2A_rC_{pr}I'_pD\gamma^2 + \alpha^2\beta^2e^{-2\beta L} + 2\alpha C_{pr}I'_pD\gamma^2e^{-\beta L} \\ &+C_{pr}I'_pDM^2\alpha\beta^2e^{-\beta L} - 2C_{pr}I'_pDM\alpha\beta\gamma e^{-\beta L}) > 0.\end{aligned} \tag{7}$$

If $T = T^*$ and $L = L^*$ be the optimal values then we must have
$\frac{\partial^2 \Pi}{\partial T^2}]_{at(T^*,L^*)} = -\frac{1}{T^{*3}}(2A_r + \alpha e^{-\beta L^*} + C_{pr}I'_pDM^2) < 0$ and
$\frac{\partial^2 \Pi}{\partial L^2}]_{at(T^*,L^*)} = -\frac{1}{T^*}(\alpha\beta^2e^{-\beta L^*} + C_{pr}I'_pD\gamma^2) < 0.$ □

Lemma 1. *When $M \leq T$ then retailer receives the order from the supplier for the next cycle $(\frac{R-DL}{D})$ times earlier if,*

$$\Big(1 - \frac{h_r}{C_{pr}I'_p\gamma}\Big)T \geq \Big[M_0 + \frac{\gamma}{2}(\frac{R}{D} - L)\Big], \; \textit{[where } h_r < C_{pr}I'_p\gamma\textit{]} \; .$$

Proof. If $\gamma \neq 0$, i.e., supplier offer extra credit period to the retailer for supplying the order $(\frac{R-DL}{D})$ times earlier, then interest paid by the retailer to the supplier is given by

$$IPC^{(\gamma \neq 0)} = C_{pr}I'_p\int_M^T(Q - Dt)dt = C_{pr}I'_p\Big[Q(T - M) - \frac{D}{2}(T^2 - M^2)\Big].$$

If $\gamma = 0$, that means if retailer does not accept the extra credit period offered by the supplier and supplier supply the order at time T then interest paid by the retailer to the supplier is given by

$$IPC^{(\gamma = 0)} = C_{pr}I'_p\int_{M_0}^T(Q - Dt)dt = C_{pr}I'_p\Big[Q(T - M_0) - \frac{D}{2}(T^2 - M_0^2)\Big].$$

The extra holding cost for the order receive ($\frac{R-DL}{D}$) times earlier for next cycle is given by

$$EHC = h_r(R-DL)T.$$

Hence the retailer receives the order from the supplier for the next cycle ($\frac{R-DL}{D}$) times earlier and takes the extra credit period opportunity from the supplier if,

$$\begin{aligned}
&IPC^{(\gamma=0)} - IPC^{(\gamma\neq 0)} \geq EHC\\
&i.e., C_{pr}I'_p\gamma(\frac{R-DL}{D})\Big[Q - \frac{D}{2}\Big\{2M_0 + \gamma\frac{R-DL}{D}\Big\}\Big] \geq h_r(R-DL)T, \text{ [by equation (1)] },\\
&i.e., (1-\frac{h_r}{C_{pr}I'_p\gamma})T \geq \Big[M_0 + \frac{\gamma}{2}(\frac{R}{D} - L)\Big], \text{ [as } Q = DT].
\end{aligned}$$

This relation valid when $(1-\frac{h_r}{C_{pr}I'_p\gamma}) > 0$ i.e., when $h_r < C_{pr}I'_p\gamma$. □

Lemma 2. *The value of T exists provided that*

$$\begin{aligned}
&(C_{pr}I'_pD\gamma - h_rD)^2 + \beta(C_{pr}I'_pD + h_rD - S_rI'_eD)\Big[C_{pr}I'_pD\beta(M_0 + \frac{\gamma R}{D} - \gamma L)^2\\
-\ &2C_{pr}I'_pD\gamma + 2A_r\beta\Big] \geq 0 \text{ and } C_{pr}I'_pD\beta\{M_0 + \frac{\gamma R}{D} - \gamma L\}^2 - 2C_{pr}I'_pD\gamma(M_0 + \frac{\gamma R}{D}\\
-\ &\gamma L) + 2A_r\beta < 0 \text{ ,where } L \in (\tfrac{M_0}{\gamma} + \tfrac{R}{D} - \tfrac{2}{\beta}, \tfrac{M_0}{\gamma} + \tfrac{R}{D}) .
\end{aligned}$$

Proof. From equation (6) we have,

$$\begin{aligned}
&\beta T^2(C_{pr}I'_pD + h_rD - S_rI'_eD) - 2(C_{pr}I'_pD\gamma - h_rD)T - \beta C_{pr}I'_pD\{(M_0 + \frac{\gamma R}{D})\\
&-\gamma L\}^2 - 2A_r\beta + 2C_{pr}I'_pD\gamma(M_0 + \frac{\gamma R}{D} - \gamma L) = 0\\
&i.e., T = \frac{Y \pm \sqrt{Y^2 + \beta X(C_{pr}I'_pD\beta Z^2 - 2C_{pr}I'_pD\gamma Z + 2A_r\beta)}}{\beta X} \qquad (8)
\end{aligned}$$

where $X = (C_{pr}I'_pD + h_rD - S_rI'_eD)$, $Y = (C_{pr}I'_pD\gamma - h_rD)$ and
$Z = (M_0 + \frac{\gamma R}{D} - \gamma L)$,(say) .

For real value of T,

$$\begin{aligned}
&(C_{pr}I'_pD\gamma - h_rD)^2 + \beta(C_{pr}I'_pD + h_rD - S_rI'_eD)\Big[C_{pr}I'_pD\beta(M_0 + \frac{\gamma R}{D} - \gamma L)^2\\
&-2C_{pr}I'_pD\gamma + 2A_r\beta\Big] \geq 0.
\end{aligned}$$

The value of T will exist if

$$C_{pr}I'_pD\beta\{M_0 + \frac{\gamma R}{D} - \gamma L\}^2 - 2C_{pr}I'_pD\gamma(M_0 + \frac{\gamma R}{D} - \gamma L) + 2A_r\beta < 0 \qquad (9)$$

[since $Y > 0$ for $h_r < C_{pr}I'_p\gamma$] .

From equation (9) we have

$$\Big[M_0 + \gamma(\frac{R-DL}{D})\Big]\Big[\beta\{M_0 + \gamma(\frac{R-DL}{D})\} - 2\gamma\Big] + \frac{2A_r\beta}{C_{pr}I'_pD} < 0.$$

Since $M_0 + \gamma(\frac{R-DL}{D}) > 0$ so, $\beta\{M_0 + \gamma(\frac{R-DL}{D})\} - 2\gamma < 0$.
Then we must have the followings

$$\frac{M_0}{\gamma} + \frac{R}{D} > L \text{ and } \frac{M_0}{\gamma} + \frac{R}{D} - \frac{2}{\beta} < L, \text{ this shows that } L \in \Big(\frac{M_0}{\gamma} + \frac{R}{D} - \frac{2}{\beta}, \frac{M_0}{\gamma} + \frac{R}{D}\Big).$$

□

3.2 When the credit period greater than to the business period $(M > T)$

Interest earned up to T is given by

$$S_r I_e' \int_0^T D(T-t)dt = \frac{S_r I_e' DT^2}{2}$$

and interest earned during $(M-T)$ i.e. beyond the cycle length and up to the permissible period is obtained as

$$S_r I_e' \int_T^M DT dt = S_r I_e' DT(M-T).$$

Hence, the total interest earned during the cycle is given by

$$\frac{S_r I_e' DT^2}{2} + S_r I_e' DT(M-T) = S_r I_e' DT(M - \frac{T}{2}).$$

Therefore, the expected average total profit $\Pi(T, L)$ is given by

$$\Pi(T,L) = \Big(S_r D + S_r I_e' D(M - \frac{T}{2})\Big) - \Big[\frac{A_r}{T} + C_{pr} D + h_r(\frac{1}{2}DT + R - DL) + \frac{\alpha e^{-\beta L}}{T}\Big]. \tag{10}$$

Theorem 2. *If $M > T$, the profit function $\Pi(T, L)$ will be optimal for optimum value of $T = T^*$ and $L = L^*$ satisfying the relations*

$$\beta(S_r I_e' D + h_r D)T^2 - 2(S_r I_e' D\gamma - h_r D)T - 2A_r\beta = 0 \ \text{and} \ \frac{\alpha\beta^2 e^{-\beta L}}{T^4}[2A_r + \alpha e^{-\beta L}] > 0,$$

then $\Pi(T^*, L^*)$ attains global optimum value.

Proof. Here the objective function of the model is given by

$$\Pi(T,L) = \Big(S_r D + S_r I_e' D(M - \frac{T}{2})\Big) - \Big[\frac{A_r}{T} + C_{pr} D + h_r(\frac{1}{2}DT + R - DL) + \frac{\alpha e^{-\beta L}}{T}\Big].$$

For optimal value of T and L, $\frac{\partial \Pi}{\partial T} = 0$ and $\frac{\partial \Pi}{\partial L} = 0$ give the following relations

$$\frac{\alpha e^{-\beta L}}{T} = T\Big(\frac{S_r I_e' D}{2} + \frac{h_r D}{2}\Big) - \frac{A_r}{T} \tag{11}$$

$$\text{and} \quad -S_r I_e' D\gamma + h_r D + \frac{\alpha\beta e^{-\beta L}}{T} = 0. \tag{12}$$

Eliminating $\frac{\alpha e^{-\beta L}}{T}$ from equation (11) and (12)

$$\beta(S_r I_e' D + h_r D)T^2 - 2(S_r I_e' D\gamma - h_r D)T - 2A_r\beta = 0 \tag{13}$$

Also,

$$(\frac{\partial^2 \Pi}{\partial T^2})(\frac{\partial^2 \Pi}{\partial L^2}) - (\frac{\partial^2 \Pi}{\partial T \partial L})^2 > 0 \text{ gives}$$

$$\frac{\alpha\beta^2 e^{-\beta L}}{T^4}[2A_r + \alpha e^{-\beta L}] > 0 \tag{14}$$

If $T = T^*$ and $L = L^*$ be the optimum values then we must have
$\frac{\partial^2 \Pi}{\partial T^2}]_{at(T^*,L^*)} = -\frac{2}{T^{*3}}(A_r + \alpha e^{-\beta L^*}) < 0$ and $\frac{\partial^2 \Pi}{\partial L^2}]_{at(T^*,L^*)} = -\frac{\alpha\beta^2 e^{-\beta L^*}}{T^*} < 0.$ □

Lemma 3. *When $M > T$ and $M_0 \leq T$, then the retailer receive the order from supplier for the next cycle $(\frac{R-DL}{D})$ times earlier if,*

$$(S_r I_e'\gamma - h_r)(R - DL)T \geq (T - M_0)D\Big[\Big(\frac{T + M_0}{2}\Big)(S_r I_e' + C_{pr} I_p') - C_{pr} I_p' T\Big],$$

where $h_r < S_r I_e'\gamma$.

Proof. If $\gamma \neq 0$, i.e., supplier offer extra credit period to the retailer for supplying the order $(\frac{R-DL}{D})$ times earlier, then interest earned by the retailer to the supplier is given by

$$IE^{(\gamma\neq 0)} = S_r I'_e \int_0^T D(T-t)dt + S_r I'_e \int_T^M DTdt = S_r I'_e DT\Big(M - \frac{T}{2}\Big).$$

If $\gamma = 0$, that means if retailer does not accept the extra credit period offered by the supplier and supplier supply the order at time T then interest earned by the retailer to the supplier is given by

$$\begin{aligned} IE^{(\gamma=0)} &= S_r I'_e \int_0^{M_0} D(T-t)dt - C_{pr} I'_p \int_{M_0}^T (Q-Dt)dt \\ &= S_r I'_e D\Big(TM_0 - \frac{M_0^2}{2}\Big) - C_{pr} I'_p \Big\{Q(T-M_0) - \frac{D}{2}(T^2 - M_0^2)\Big\}. \end{aligned}$$

The extra holding cost for order receive $(\frac{R-DL}{D})$ times earlier (for next cycle) is given by

$$EHC = h_r(R-DL)T.$$

Therefore, the retailer receive the order from the supplier for the next cycle $(\frac{R-DL}{D})$ times earlier and takes extra credit period opportunity from the supplier if,

$$\begin{aligned} & S_r I'_e D\Big[TM_0 + \gamma\Big(\frac{R-DL}{D}\Big)T - TM_0 - \frac{1}{2}(T_p^2 - M_0^2)\Big] + C_{pr} I'_p \Big[Q(T-M_0)\Big] \\ - \quad & \frac{D}{2}(T^2 - M_0^2) \geq h_r(R-DL)T, \text{ Since } M = M_0 + \gamma\Big(\frac{R-DL}{D}\Big) \\ & i.e., (S_r I'_e \gamma - h_r)(R-DL)T \geq (T-M_0)D\Big[\Big(\frac{T+M_0}{2}\Big)(S_r I'_e + C_{pr} I'_p) - C_{pr} I'_p T\Big]. \end{aligned}$$

Since $(R-DL) > 0$, the relation holds when $(S_r I'_e \gamma - h_r) > 0$, i.e., when $h_r < S_r I'_e \gamma$. □

Lemma 4. *When $M > T$ and $M_0 > T$, then the retailer receive the order from supplier for the next cycle $(\frac{R-DL}{D})$ times earlier if,*

$$h_r \leq S_r I'_e \gamma.$$

Proof. If $\gamma \neq 0$, i.e., supplier offer extra credit period to the retailer for supplying the order $(\frac{R-DL}{D})$ times earlier, then interest earned by the retailer to the supplier is given by

$$IE^{\gamma\neq 0} = S_r I'_e \int_0^T D(T-t)dt + S_r I'_e \int_T^M DTdt = S_r I'_e DT\Big(M - \frac{T}{2}\Big).$$

If $\gamma = 0$, that means if retailer does not accept the extra credit period offered by the supplier and supplier supply the order at time T then interest earned by the retailer to the supplier is given by

$$IE^{\gamma=0} = S_r I'_e \int_0^T D(T-t)dt + S_r I'_e \int_T^{M_0} DTdt = S_r I'_e DT\Big(M_0 - \frac{T}{2}\Big).$$

The extra holding cost for order receive $(\frac{R-DL}{D})$ times earlier (for next cycle) is given by

$$EHC = h_r(R-DL)T.$$

Hence, the retailer receive the order from the supplier for the next cycle $(\frac{R-DL}{D})$ times earlier and takes extra credit period opportunity from the supplier if,

$$\begin{aligned} & S_r I'_e DT\Big[M_0 + \gamma\Big(\frac{R-DL}{D}\Big) - M_0\Big] \geq h_r(R-DL)T, \text{ as } M = M_0 + \gamma(\tfrac{R-DL}{D}) \,, \\ & i.e., (1 - \frac{h_r}{S_r I'_e \gamma}) \geq 0, \text{ since } (R-DL) > 0 \\ & i.e., h_r \leq S_r I'_e \gamma. \end{aligned}$$

□

Lemma 5. *The value of T exists provided that*

$$(S_rI'_e\gamma - h_r)^2 + \Big(\frac{2A_r\beta^2}{D}\Big)(S_rI'_e + h_r) \geq 0 \quad \textit{and} \quad \Big(\frac{2A_r\beta^2}{D}\Big)(S_rI'_e + h_r) > 0.$$

Proof. From equation (13) we have,

$$\begin{aligned} &2(h_rD - S_rI'_eD\gamma)T + \beta T^2(S_rI'_eD + h_rD) - 2A_r\beta = 0, \\ &i.e., T = \frac{(S_rI'_e\gamma - h_r) \pm \sqrt{(S_rI'_e\gamma - h_r)^2 + (\frac{2A_r\beta^2}{D})(S_rI'_e + h_r)}}{\beta(S_rI'_e + h_r)}. \end{aligned}$$

For all real value of T,

$$(S_rI'_e\gamma - h_r)^2 + \Big(\frac{2A_r\beta^2}{D}\Big)(S_rI'_e + h_r) \geq 0.$$

Since $h_r \leq S_rI'_e\gamma$ and $h_r < S_rI'_e\gamma$, according to Lemma-3.3 and Lemma-3.4 respectively, therefore $(S_rI'_e\gamma - h_r) > 0$.
Therefore, the value of T exists if, $(\frac{2A_r\beta^2}{D}).(S_rI'_e + h_r) > 0$. Therefore

$$T = \frac{(S_rI'_e\gamma - h_r) - \sqrt{(S_rI'_e\gamma - h_r)^2 + (\frac{2A_r\beta^2}{D})(S_rI'_e + h_r)}}{\beta(S_rI'_e + h_r)}. \tag{15}$$

□

4 Solution Procedure

The objective function of the proposed model $\Pi(T, L)$ is highly nonlinear. Here T and L are two decision variables. Since the objective function is highly nonlinear, hence to get the optimal solution of the proposed model the following algorithm have been developed.

Algorithm 4.1. *When the credit period be less than or equal to the business period ($M \leq T$), then there is no possibility to get the general explicit solution due to absence of linearity of the profit function. Here, T is a function of L according to equation (8). So, to get the maximum profit, the following procedure has been devised according to Lemma 3.1, Lemma 3.2. Here the optimal values of T, L and $\Pi(T, L)$ are denoted by T^*, L^* and $\Pi(T^*, L^*)$ respectively.*

Step-1: Initialize all parameters associated with the objective function $\Pi(T, L)$.

Step-2: Compute the values

$$\psi_1 = \beta\Big\{M_0 + \gamma\Big(\frac{R - DL}{D}\Big)\Big\} - 2\gamma, \quad \psi_2 = M_0 + \gamma\Big(\frac{R - DL}{D}\Big) \textit{and} \ \ \psi = \psi_1\psi_2 + \frac{2A_r\beta}{C_{pr}I'_pD};$$

Step-3: If $\psi \geq 0$, then goto Step-1;

Step-4: Compute the interval (L_0, L_1) in which L belongs to, where $L_0 = \frac{M_0}{\gamma} + \frac{R}{D} - \frac{2}{\beta}$ and $L_1 = \frac{M_0}{\gamma} + \frac{R}{D}$;

Step-5: Set the value ϵ, then compute the value of $L = L^*$ in (L_0, L_1) and say $L^* = L_0 + \epsilon$;

Step-6: For $L = L^*$, compute the value of T (say T^*), where

$$T^* = \frac{(C_{pr}I'_pD\gamma - h_rD) - \sqrt{(C_{pr}I'_pD\gamma - h_rD)^2 + \beta(C_{pr}I'_pD + h_rD - S_rI'_eD)\{C_{pr}I'_pD\beta(M_0 + \frac{\gamma R}{D} - \gamma L^*)^2 - (M_0 + \frac{\gamma R}{D} - \gamma L^*) - 2A_r\beta\}}}{\beta(C_{pr}I'_pD + h_rD - S_rI'_eD)};$$

*Step-*7*: Compute the values of* $\frac{\partial^2\Pi(T,L)}{\partial T^2}$, $\frac{\partial^2\Pi(T,L)}{\partial L^2}$ *and* $\frac{\partial^2\Pi(T,L)}{\partial T\partial L}$ *and let*

$$\Delta_1 = \frac{\partial^2\Pi(T,L)}{\partial T^2},\ \ \Delta_2 = \frac{\partial^2\Pi(T,L)}{\partial L^2} \text{and}\ \ \Delta_3 = \frac{\partial^2\Pi(T,L)}{\partial T\partial L};$$

*Step-*8*: If* $\Delta_1 < 0$, $\Delta_2 < 0$ *and* $\Delta_1\Delta_2 > (\Delta_3)^2$, *then* (T^*, L^*) *is the optimal solution. Also calculate the value of expected average profit* $\Pi(T^*, L^*)$ *and goto Step-*11*;*

*Step-*9*: If* $\Delta_1 < 0$, $\Delta_2 > 0$ *or,* $\Delta_1 > 0$, $\Delta_2 < 0$ *or,* $\Delta_1 > 0$, $\Delta_2 > 0$ *then* (T^*, L^*) *is not an optimal solution. Then goto Step-*10*;*

*Step-*10*: Compute* $L^* = L^* + \epsilon$, *then goto Step-*6*;*

*Step-*11*: Print the optimal values* T^*, L^* *and* $\Pi(T^*, L^*)$.

Algorithm 4.2. *When the credit period greater than to the business period* $(M > T)$*, then there is no possibility to get the general explicit solution due to absence of linearity of the profit function. Here,* L *is a function of* T *according to equation* (11)*. So, to get the maximum profit, the following procedure has been devised according to Lemma* 3.3*, Lemma* 3.4*, Lemma* 3.5*. Here the optimal values of* T*,* L *and* $\Pi(T, L)$ *are denoted by* T^**,* L^* *and* $\Pi(T^*, L^*)$ *respectively.*

*Step-*1*: Initialize all parameters associated with the objective function* $\Pi(T, L)$.

*Step-*2*: Compute the values*

$$\psi_1 = (S_r I'_e\gamma - h_r)^2 + \frac{2A_r\beta^2}{D}(S_r I'_e + h_r),$$
$$\text{and } \psi_2 = (S_r I'_e\gamma - h_r) - \sqrt{(S_r I'_e\gamma - h_r)^2 + \frac{2A_r\beta^2}{D}(S_r I'_e + h_r)}.$$

*Step-*3*: If* $\psi_1, \psi_2 < 0$ *then goto step-*1.

*Step-*4*: Compute the value* $T = T^*$ *from equation* (15).

*Step-*5*: Compute the value* $L = L^*$ *from the equation* (11) *by using the value of* $T = T^*$.

*Step-*6*: Compute the value of* $\frac{\partial^2\Pi(T,L)}{\partial T^2}$, $\frac{\partial^2\Pi(T,L)}{\partial L^2}$ *and* $\frac{\partial^2\Pi(T,L)}{\partial T\partial L}$ *and let*

$$\Delta_1 = \frac{\partial^2\Pi(T,L)}{\partial T^2},\ \ \Delta_2 = \frac{\partial^2\Pi(T,L)}{\partial L^2} \text{and}\ \ \Delta_3 = \frac{\partial^2\Pi(T,L)}{\partial T\partial L}.$$

*Step-*7*: If* $\Delta_1 < 0$, $\Delta_2 < 0$ *and* $\Delta_1\Delta_2 > (\Delta_3)^2$, *then* (T^*, L^*) *is the optimal solution. Also calculate the value of expected average profit* $\Pi(T^*, L^*)$ *and goto Step-*9.

*Step-*8*: If* $\Delta_1 < 0$, $\Delta_2 > 0$ *or,* $\Delta_1 > 0$, $\Delta_2 < 0$ *or,* $\Delta_1 > 0$, $\Delta_2 > 0$ *then* (T^*, L^*) *is not an optimal solution. Then goto Step-*4 *and changes some parametric values.*

*Step-*9*: Print the optimal values* T^*, L^* *and* $\Pi(T^*, L^*)$.

5 Numerical examples

Some developing countries like India and China, offers credit period to the retailer, to extend their business. These types of business are available for the electronic goods, like air conditioners, mobiles, washing machines etc and also for useful things in every day life like water filter. Considering these kinds of phenomena, some examples have been considered here.

5.1 When the credit period to be less than or equal to the business period ($M \leq T$)

A company supplies the items (water filter) to the retailer at cost \$ 40 per unit item and offers a credit period 2 month to a retailer at a condition that he has to pay the total purchasing cost at the end of the business cycle and he will enjoy the relaxation of the credit period for variability of lead time. Retailer's holding cost is \$ 2 per unit item per unit time and ordering cost is \$ 400 per order. Retailer sales each item at a price \$ 55 and per month customer demand rate is 70 unit. Retailer earns the interest at the rate 0.06 per month and pays the interest payable at the rate 0.08 per month. He orders again when his stock becomes 110 unit. His objective is to maximize the average total profit. Find the optimal values of business period, lead time, the order quantity and offered credit period by the supplier to the retailer.

Solution. In this inventory system, the following parameters are: A_r=\$ 400 per order, C_{pr}=\$ 40 per item, S_r=\$ 55 per item, D= 70 unit/month, $I'p = 0.08$/month, $I'e = 0.06$/month, $M_0 = 2$ month, $\alpha = 46$, $\beta = 1$, $\gamma = 4$, h_r=\$ 2 per item per unit time, $R = 110$ unit.

Numerically for Example 5.1, Fig.2 shows the graphical representation of the average profit

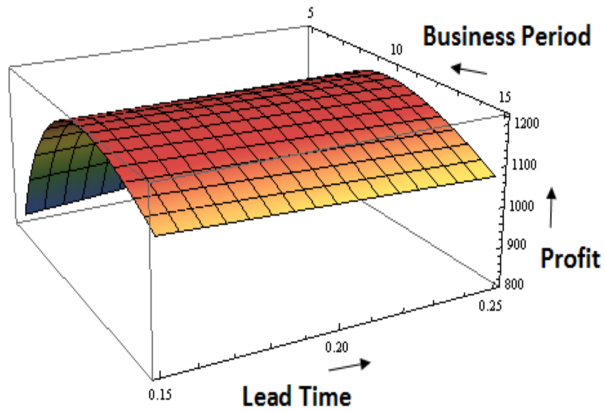

Figure 2. Concavity of the Profit function $\Pi(T, L)$

function of T and L. From this figure, it is guarantied that the profit function $\Pi(T, L)$ is concave. So there exists a solution of (T, L) that maximizes the average total profit $\Pi(T, L)$.
According to the Algorithm 4.1 in Section 4 and above parametric values we have the following optimal solutions:
offered credit period by the supplier to the retailer $(M^*) = 7.525418$ month, the order quantity $(Q^*) = 706.8513$ unit, lead time $(L^*) = 0.1900741$ month, the business period $(T^*) = 10.09788$ month and expected average total profit $\Pi(T^*, L^*)$=\$ 1199.29.

5.1.1 Results of effective parameters

Now, we examine the effects of the system parameters α, β and γ on business cycle period, lead time, the order quantity and offered credit period by the supplier to the retailer numerically considering Example 5.1 as follows:

Table 2: Optimum results for different values of α

α	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
44	7.525418	0.1900741	706.7651	10.09664	\$ 1199.45
45	7.525418	0.1900741	706.8082	10.09726	\$ 1199.37
46	7.525418	0.1900741	706.8513	10.09788	\$ 1199.29
47	7.525418	0.1900741	706.8944	10.09849	\$ 1199.21
48	7.525418	0.1900741	706.9375	10.09911	\$ 1199.12

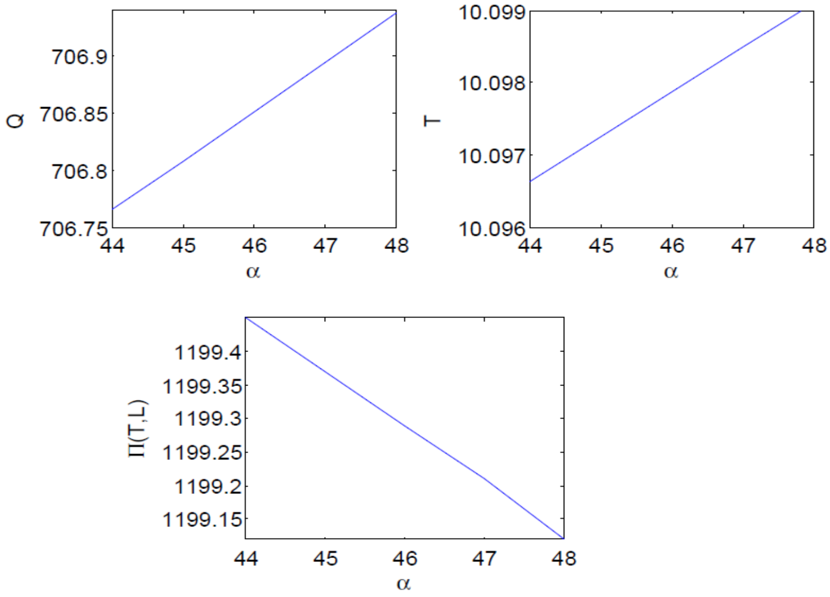

Figure 3. Effects of Q, T, $\Pi(T, L)$ for different values of α

From Fig.3, it is observed that the order quantity (Q) and business period (T) are increases for increasing value of α in $[44, 48]$. It is also seen that the value of average total profit $\Pi(T, L)$ decreases for increasing value of α in $[44, 48]$ and there are no effects of α on the offered credit period by the supplier to the retailer (M) and lead time (L).

Table 3: Optimum results for different values of β

β	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
1.0	7.525418	0.1900741	706.851	10.09788	\$ 1199.29
1.1	6.743084	0.3856576	637.912	9.113029	\$ 1182.41
1.2	6.079182	0.5516332	579.836	8.283375	\$ 1167.28
1.3	5.505097	0.6951544	530.049	7.572125	\$ 1153.37
1.4	5.000000	0.8214286	486.685	6.952649	\$ 1140.30

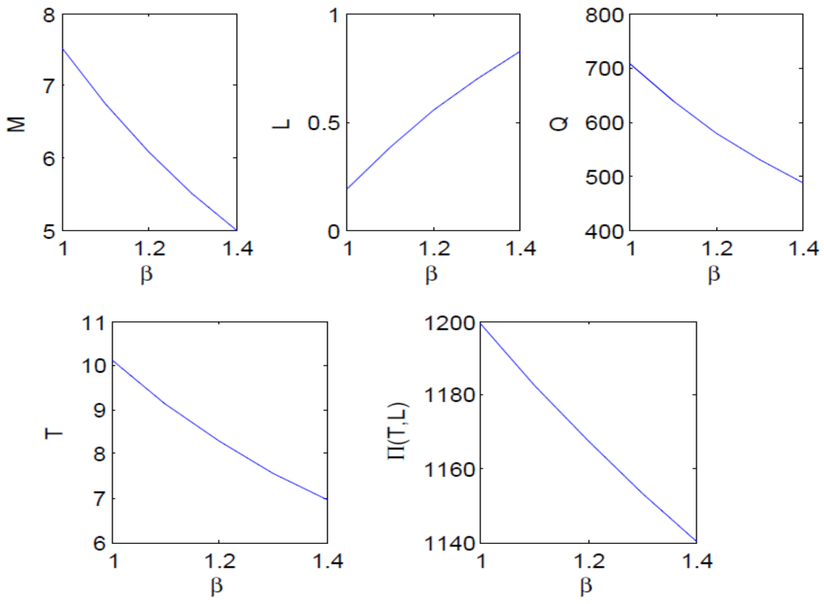

Figure 4. Changes of M, L, Q, T, $\Pi(T, L)$ for different values of β

In Fig.4, it is observed that the values of lead time (L) increases but the values of offered credit period by the supplier to the retailer (M), ordering quantity (Q), business period (T) and average total profit $\Pi(T, L)$ are decreases with respect to the increasing value of β in $[1.0, 1.4]$.

Table 4: Optimum results for different values of γ

γ	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
3.8	7.096752	0.2301781	669.1792	9.559703	\$ 1180.46
3.9	7.311535	0.2094965	688.0379	9.829113	\$ 1189.84
4.0	7.525418	0.1900741	706.8513	10.09788	\$ 1199.29
4.1	7.738485	0.1717981	725.6238	10.36605	\$ 1208.79
4.2	7.950809	0.1545692	744.3594	10.63371	\$ 1218.34

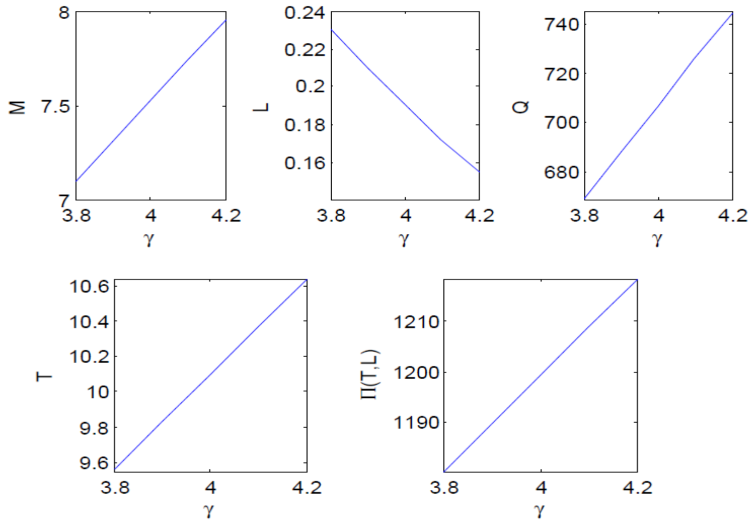

Figure 5. Changes of M, L, Q, T, $\Pi(T, L)$ for different values of γ

In Fig.5, when the values of γ increases in $[3.8, 4.2]$, the values of offered credit period by the supplier to the retailer (M), order quantity (Q), business period (T) and average total profit $\Pi(T, L)$ are also increases but the values of lead time (L) decreases.

5.2 When the credit period to be greater than the business period (i.e., $M > T$ and $M_0 < T$)

In this example, using the same data as in Example 5.1 *except cost of the item \$* 10 *per unit item, selling price \$* 20 *per unit item and per month customer demand rate is* 70 *unit, retailer order again when his stock become* 80 *unit,* $\alpha = 85$ *and* $\gamma = 2$.

Solution. In this inventory system, the following parameters are: A_r=\$ 400 per order, C_{pr}=\$ 10 per item, S_r=\$ 20 per item, D= 70 unit/month, $I'p = 0.08$/month, $I'e = 0.06$/month, $M_0 = 2$ month, $\alpha = 85$, $\beta = 1$, $\gamma = 2$, h_r=\$ 2 per item per unit time, $R = 80$ unit. Numerically for Example 5.2, Fig.6 shows the graphical representation of the average profit function of T and L. From this figure, it is guarantied that the profit function $\Pi(T, L)$ is concave. So there exists a solution of (T, L) that maximizes the average total profit $\Pi(T, L)$.

According to the Algorithm 4.2 in Section 4 and above parametric values we have the following optimal solutions:
offered credit period by the supplier to the retailer $(M^*) = 3.469978$ month, the order quantity $(Q^*) = 141.3266$ unit, lead time $(L^*) = 0.4078683$ month, the business period $(T^*) = 2.018952$ month and expected average total profit $\Pi(T^*, L^*)$=\$ 436.33.

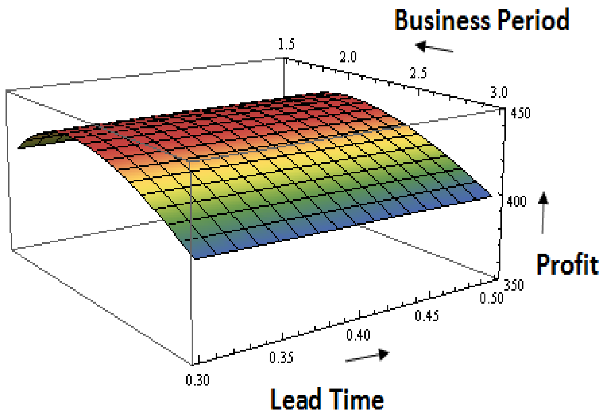

Figure 6. Concavity of the Profit function $\Pi(T, L)$

5.2.1 Results of effective parameters

Now, we examine the effects of the system parameters α, β and γ on the business period, lead time, order quantity and offered credit period by the supplier to the retailer numerically considering Example 5.2 *as follows:*

Table 5: Optimum results for different values of α

α	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
85.0	3.469978	0.4078683	141.3266	2.018952	\$ 436.33
85.1	3.467626	0.4090440	141.3266	2.018952	\$ 436.30
85.2	3.465277	0.4102184	141.3266	2.018952	\$ 436.27
85.3	3.462931	0.4113915	141.3266	2.018952	\$ 436.24
85.4	3.460588	0.4125631	141.3266	2.018952	\$ 436.20

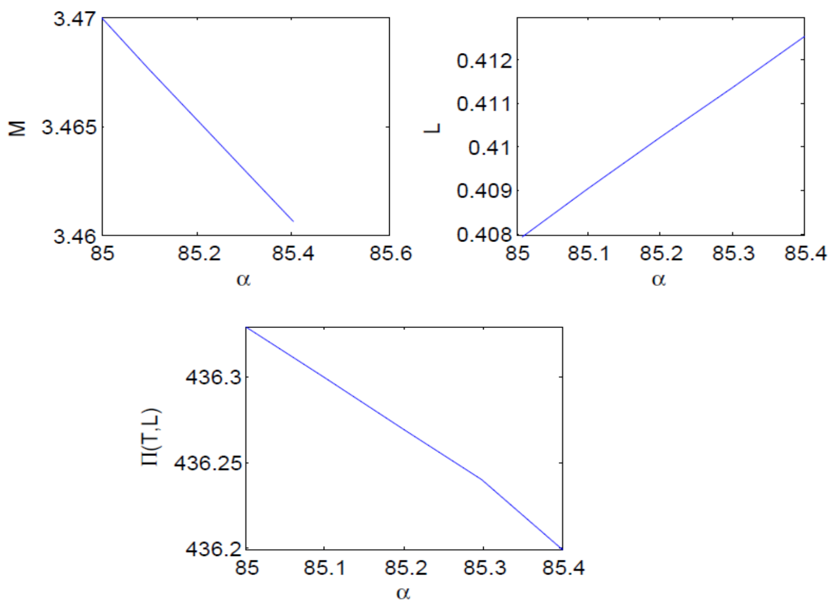

Figure 7. Changes of M, L, $\Pi(T, L)$ for different values of α

In Fig.7, the value of lead time (L) increases when the value of α increases in $[85.0, 85.4]$. The values of offered credit period by the supplier to the retailer (M) and average total profit

$\Pi(T, L)$ decreases for increasing value of α in $[85.0, 85.4]$ and there are no effects of α on the order quantity (Q) and business period (T).

Table 6: Optimum results for different values of β

β	Offered credit period (M^*) month	Lead time (L^*) month	order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
0.7	4.220177	0.0327684	145.3768	2.076812	\$ 433.88
0.8	3.865128	0.2102932	143.6765	2.052521	\$ 434.35
0.9	3.629767	0.3279737	142.3666	2.033808	\$ 435.24
1.0	3.469978	0.4078683	141.3266	2.018952	\$ 436.33
1.1	3.359933	0.4628905	140.4811	2.006872	\$ 437.50

In Fig.8, for increasing values of β in $[0.7, 1.1]$, the values of lead time (L) and average total profit $\Pi(T, L)$ are increases and the values of offered credit period by the supplier to the retailer (M), order quantity (Q) and business period (T) are decreases.

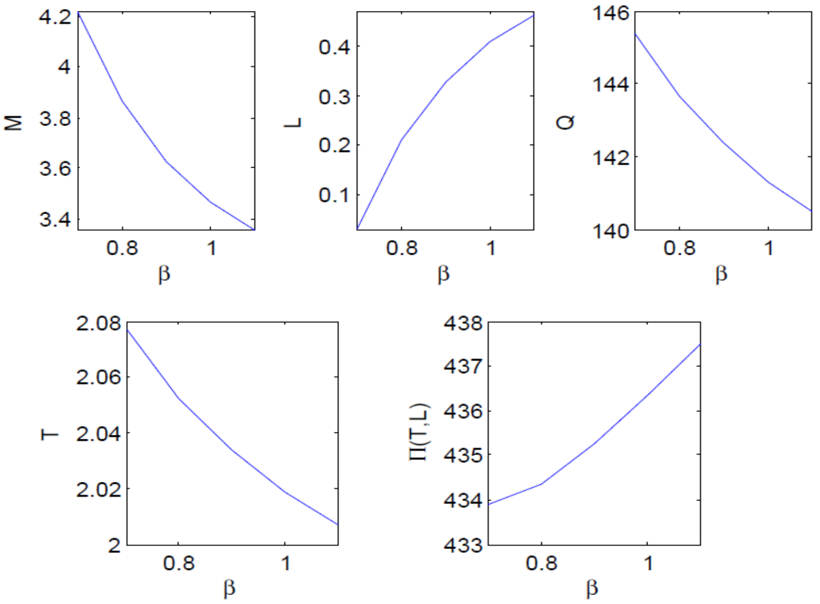

Figure 8. Changes of M, L, Q, T, $\Pi(T, L)$ for different values of β

Table 7: Optimum results for different values of γ

γ	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
2.01	3.540720	0.3763296	141.6067	2.022953	\$ 436.97
2.02	3.610379	0.3456399	141.8873	2.026962	\$ 437.62
2.03	3.679024	0.3157518	142.1684	2.030978	\$ 438.30
2.04	3.746720	0.2866218	142.4501	2.035001	\$ 439.01
2.05	3.813527	0.2582098	142.7322	2.039032	\$ 439.74

In Fig.9, when the values of γ increases in $[2.01, 2.05]$, the values of offered credit period by the supplier to the retailer (M), order quantity (Q), business period (T) and average total profit $\Pi(T, L)$ are increases but the value of lead time (L) decreases.

5.3 When the credit period to be greater than the business period i.e., ($M > T$ and $M_0 > T$)

In this example, using the same data as in Example 5.1 *except the cost of the item \$* 10 *per unit item, selling price \$* 20 *per unit item and per month customer demand rate is* 30 *unit, retailer*

order again when his stock become 40 *unit. Retailer's earned interest at the rate* 0.16 *per month and interest payable at the rate* 0.18 *per month, offers a credit period* 5 *month to retailer at a condition that he has to pay the total purchase cost at the end of the business period and he will enjoy the relaxation of the credit period for variability of lead time.* $\alpha = 156$, $\beta = 3$ *and* $\gamma = 2$.
Solution In this inventory system, the following parameters are: A_r=\$ 400 per order, C_{pr}=\$ 10 per item, S_r=\$ 20 per item, D=30 unit/month, $I'p = 0.18$/month, $I'e = 0.16$/month, $M_0 = 5$ month, $\alpha = 156$, $\beta = 3$, $\gamma = 2$, h_r=\$ 2 per item per unit time, $R = 40$ unit.

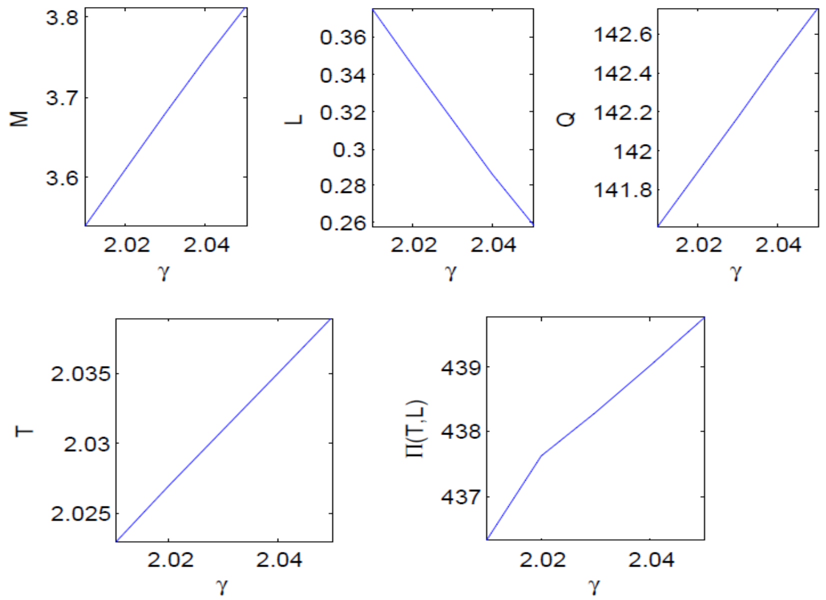

Figure 9. Changes of M, L, Q, T, $\Pi(T, L)$ for different values of γ

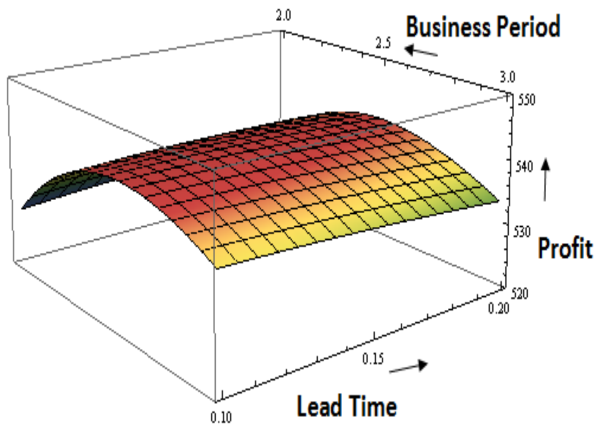

Figure 10 Concavity of the Profit function $\Pi(T, L)$

Numerically for Example 5.3, Fig.10 shows the graphical representation of the average profit function of T and L. From this figure, it is guarantied that the average profit function $\Pi(T, L)$ is concave. So there exists a solution of (T, L) that maximizes the average total profit $\Pi(T, L)$. According to the Algorithm 4.2 in Section 4 and above parametric values we have the following optimal solutions:
offered credit period by the supplier to the retailer $(M^*) = 7.450628$ month, the order quantity $(Q^*) = 76.92308$ unit, lead time $(L^*) = 0.1080193$ month, the business period $(T^*) = 2.564103$ month and expected average total profit $\Pi(T^*, L^*)$=\$ 541.74.

5.3.1 Results of effective parameters

Now, we examine the effects of the system parameters α, β and γ on the business period, lead time, order quantity and offered credit period by the supplier to the retailer numerically considering Example 5.3 *as follows:*

Table 8: Optimum results for different values of α

α	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
156.0	7.450628	0.1080193	76.92308	2.564103	$ 541.74
156.1	7.450201	0.1082329	76.92308	2.564103	$ 541.71
156.2	7.449774	0.1084464	76.92308	2.564103	$ 541.68
156.3	7.449347	0.1086597	76.92308	2.564103	$ 541.66
156.4	7.448921	0.1088729	76.92308	2.564103	$ 541.63

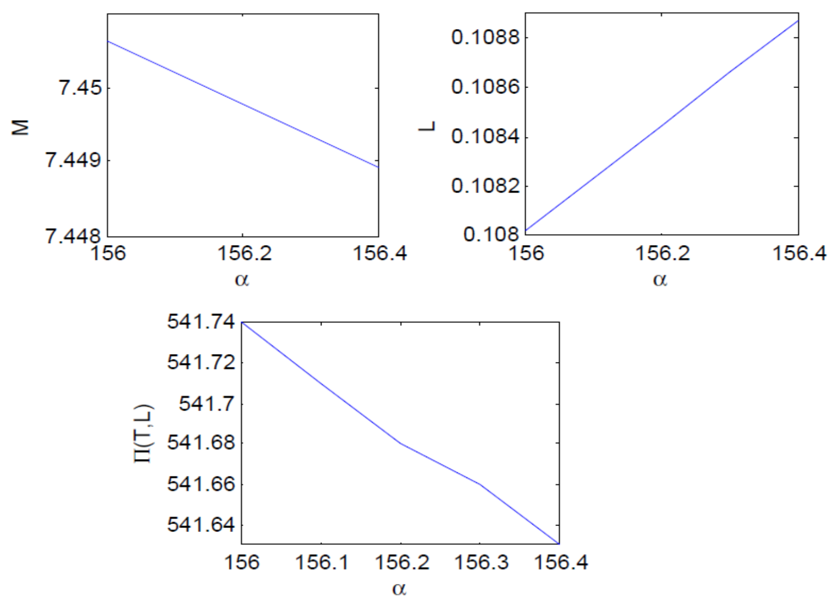

Figure 11. Changes of M, L, $\Pi(T, L)$ for different values of α

In Fig.11, for different values of α in $[156.0, 156.4]$, the values of lead time (L) increases and offered credit period by the supplier to the retailer (M) and average total profit $\Pi(T, L)$ are decreases. But there are no effects of α on the order quantity (Q) and business period (T).

Table 9: Optimum results for different values of β

β	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
3.0	7.450628	0.1080193	76.92308	2.564103	$ 541.74
3.1	7.433870	0.1163986	76.61692	2.553897	$ 542.23
3.2	7.418964	0.1238515	76.33088	2.544363	$ 542.73
3.3	7.405690	0.1304883	76.06303	2.535434	$ 543.25
3.4	7.393858	0.1364042	75.81170	2.527057	$ 543.77

In Fig.12, the values of lead time (L) and average total profit $\Pi(T, L)$ are increases but the values of offered credit period by the supplier to the retailer (M), order quantity (Q) and business period (T) are decreases, with respect to increasing values of β in $[3.0, 3.4]$.

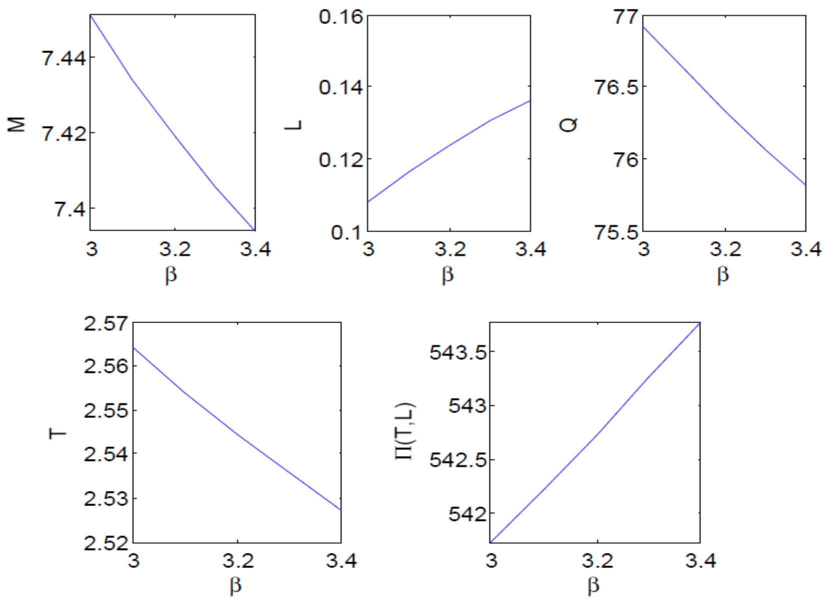

Figure 12. Changes of M, L, Q, T, $\Pi(T, L)$ for different values of β

Table 10: Optimum results for different values of γ

γ	Offered credit period (M^*) month	Lead time (L^*) month	Order quantity (Q^*) unit	Business period (T^*) month	Average total profit $\Pi(T^*, L^*)$
2.01	7.468339	0.1053042	76.99225	2.566408	\$ 542.92
2.02	7.486068	0.1026065	77.06147	2.568716	\$ 544.10
2.03	7.503817	0.0999260	77.13075	2.571025	\$ 545.28
2.04	7.521585	0.0972624	77.20009	2.573336	\$ 546.47
2.05	7.539371	0.0946155	77.26948	2.575649	\$ 547.66

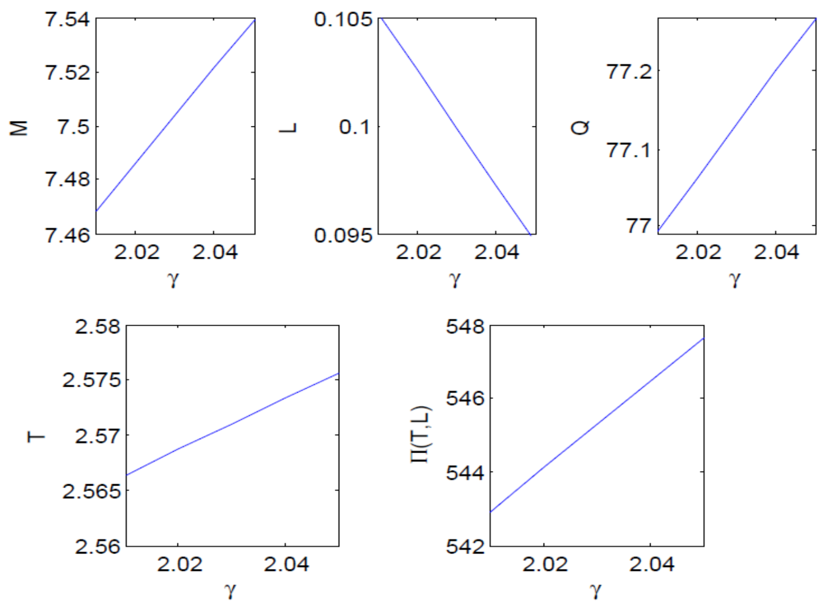

Figure 13. Changes of M, L, Q, T, $\Pi(T, L)$ for different values of γ

In Fig.13, the values of offered credit period by the supplier to the retailer (M), order quantity (Q), business period (T) and average total profit $\Pi(T, L)$ are increases but the values of lead time (L) decreases for increasing values of γ in $[2.01, 2.05]$.

Managerial implications:
From the above all sensitive analysis we have the followings managerial implications
(i). When α is increased, the value of crashing cost increases. As the value of crashing cost increases, retailer wants to bring more quantity and for more quantity, business period is also increased. As crashing cost is increased, the value of average total profit is decreased.
(ii). When the value of β is increased, the value of crashing cost is decreased. As crashing cost decreases, the value of lead time increases and for increasing value of lead time, the offered credit period by the supplier is decreased. Since offered credit period by the supplier is decreased, the value of order quantity and business period are also decreased. As the value of business period decreases, the value of average total profit also decreases.
(iii). When γ increases, the value of offered credit period by the supplier also increases. Since, offered credit period by the supplier is increased, the value of lead time is decreased. As the value of lead time decreases, average total profit of the retailer increases and for more profit retailer wants to bring more quantity. For more quantity, the value of business period also increases.
(iv). When the value of offered credit period by the supplier decreases, the value of lead time increases. As the value of lead time increases, the value of average total profit decreases.
(v). When the value of β increases, the value of crashing cost decreases. As crashing cost decreases, the value of lead time increases and for increasing value of lead time, the offered credit period by the supplier decreases. Since crashing cost decreases, the value of average total profit increases. As the supplier offers less credit period to the retailer, retailer wants to bring less quantity for that business period also decreases.

6 Conclusion

In this proposed model, an EOQ inventory model has been considered for non-deteriorating items under the joint effects of lead time and credit period as a variable in the parlance of infinite time horizon in such a way that the system gets the optimum cost. Here, a negative exponential lead time crashing cost and lead time dependent credit period offered by supplier to the retailer has been considered. Some analytical results have been drawn to get the optimal solution of the model. Two cases have been discussed depending upon the positions of the credit period and the business period. Finally, numerical feasibility of the model has been explored to get the optimum value of the profit function of the model including showing the effects of changes of some sensible parameters. From this study, it is observed that-(i) When the credit period (M) increases, the value of business period (T) is marginally increases and total cost decreases which means that profit $\Pi(T, L)$ increases significantly, (ii) The lead time (L) and business period (T) follow the inverse relation, (iii) When the lead time (L) decreases, than the credit period (M) increases, (iv) The value of lead time crashing cost $\omega(L)$ is decreasing for increasing the value of lead time (L).

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